# Toward a No-Go Theorem for an Accelerating Universe through a Nonlinear Backreaction

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The backreaction of nonlinear inhomogeneities to the cosmic expansion is re-analyzed in the framework of general relativity. Apparent discrepancies regarding the effect of the nonlinear backreaction, which exist among the results of previous works in different gauges, are resolved. By defining the spatially averaged matter energy density as a conserved quantity in the large comoving volume, it is shown that the nonlinear backreaction neither accelerates nor decelerates the cosmic expansion in a matter-dominated universe. The present result in the Newtonian gauge is consistent with the previous results obtained in the comoving synchronous gauge. Although our work does not give a complete proof, it strongly suggests the following no-go theorem: No cosmic acceleration occurs as a result of the nonlinear backreaction via averaging.

The recent observation of the isotropy of the cosmic microwave background radiation (CMBR)[1)] and large galaxy surveys, such as SDSS,[2)] indicate that the universe is remarkably isotropic and homogeneous over scales larger than some 100 Mpc. However, it is not straightforward to describe the universe using an isotropic and homogeneous metric, namely, the Friedmann-Lemaitre-Robertson-Walker (FLRW) model, because the local universe is in fact very inhomogeneous. The solution of the Einstein equation with an averaged homogeneous matter distribution is not a solution with a realistic matter distribution, because of the nonlinearity of the Einstein equation. Thus, it is naturally conjectured that the expansion law of the FLRW model may be modified by local inhomogeneities. In fact, there have been investigations studying this point,[3),4),5),6),7),8),9),10),11),12)] and some modification has been reported, with apparent discrepancies among these works (for instance, see Refs. 5) and 10)). The result might depend on the choice of the coordinates as well as the definition of the averaging procedure. For these reasons, it has not been possible to clearly relate such a nonlinear effect with observations. Furthermore, recent observations of Type Ia supernovae[13),14)] and the CMBR[15),16)] strongly suggest that the cosmic expansion is accelerating. Understanding the source of this accelerated expansion is one of the greatest unsolved problems in modern cosmology.[17),18)] This acceleration seems to require an unknown type of energy (dark energy, or perhaps a cosmological constant). A possible alternative idea to explain the acceleration is that the energy resulting inhomogeneities leads to additional terms in the Friedmann equation, as if dark energy existed.[19),20)] Before reaching a definite conclusion re-

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garding effects due to the nonlinear backreaction, we must carefully elucidate the construction of a FLRW model with a mean density which obeys the equation of state (EOS) for dust matter. Otherwise, we would be misled to conclude that a "correction" appears as a deviation from an averaged FLRW model, although the averaged matter energy density in such a model does not necessarily satisfy the EOS for dust matter.

Here we attempt to clear up this confusion by carefully constructing an averaged FLRW model in the framework of general relativity. Because there is no unique choice of the averaged spacetime in the inhomogeneous universe, we take the point of view that the averaged density of the matter is conserved in a large comoving volume. Our choice seems natural and the most suitable for describing the averaged FLRW model with a mean density that satisfies the EOS for dust matter. The purpose of this Letter is to show that the nonlinear backreaction neither accelerates nor decelerates the cosmic expansion in a matter-dominated universe. It is also shown that the backreaction behaves like a small positive curvature term. We show that the conclusion does not depend on the definition of the averaging procedure.

We restrict out study to the case of a vanishing shift vector, i.e., $N^i = 0$. The line element is written as

$$ds^2 = -(Ndt)^2 + \gamma_{ij} dx^i dx^j. \tag{1}$$

The unit vector normal to a hypersurface foliated by $t = \text{const}$ is denoted by

$$n^\mu = \left( \frac{1}{N}, 0, 0, 0 \right). \tag{2}$$

The extrinsic curvature is defined as

$$K^i{}_j \equiv \frac{1}{2N} \gamma^{ik} \dot{\gamma}_{kj}, \tag{3}$$

where the dot denotes differentiation with respect to time. The Einstein equation is reduced to the Hamiltonian constraint, the momentum constraint and (the trace of) the evolution equation as[21)]

$${}^{(3)}R + \left(K^i{}_i\right)^2 - K^i{}_j K^j{}_i = 16\pi G E, \tag{4}$$

$$K^j{}_{j|i} - K^j{}_{i|j} = 8\pi G J_i, \tag{5}$$

$$\dot{K}^i{}_i + N K^i{}_j K^j{}_i - N^{|i}{}_{|i} = -4\pi G N (E + S), \tag{6}$$

where ${}^{(3)}R$ is the 3-dimensional Ricci scalar curvature, $|$ represents the 3-dimensional covariant derivative, and we have

$$E = T_{\mu\nu} n^\mu n^\nu = \frac{1}{N^2} T_{00}, \tag{7}$$

$$J_i = -T_{\mu i} n^\mu = \frac{1}{N} T_{0i}, \tag{8}$$

$$S = T_{ij} \gamma^{ij}. \tag{9}$$

The three-dimensional volume $V$ of a compact domain $D$ on a $t=$ const hypersurface is

$$V=\int_D \sqrt{\gamma}\, d^3x, \tag{10}$$

where

$$\gamma=\det(\gamma_{ij}). \tag{11}$$

Here, $V$ is considered to be a volume sufficiently large that we can assume periodic boundary conditions.

The scale factor $a(t)$ is defined from the volume expansion rate of the universe:[6),7)]

$$3\frac{\dot{a}}{a}\equiv\frac{\dot{V}}{V}=\frac{1}{V}\int_D \frac{1}{2}\gamma^{ij}\,\dot{\gamma}_{kj}\sqrt{\gamma}\, d^3x. \tag{12}$$

Next, we introduce the averaging procedure[7)]

$$\langle A\rangle\equiv\frac{1}{V}\int_D A\sqrt{\gamma}\, d^3x. \tag{13}$$

With this, we find

$$3\frac{\dot{a}}{a}=\langle NK^i{}_i\rangle. \tag{14}$$

Then we define $V^i_j$ as

$$V^i_j\equiv NK^i{}_j-\frac{\dot{a}}{a}\delta^i{}_j, \tag{15}$$

which represents the deviation from uniform Hubble flow. With these definitions, one can show $\langle V^i_i\rangle=0$.

Averaging the Einstein equations, we obtain

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\langle N^2E\rangle-\frac{1}{6}\langle N^{2\,(3)}R\rangle-\frac{1}{6}\langle (V^i_i)^2-V^i_jV^j_i\rangle, \tag{16}$$

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}\langle N^2(E+S)\rangle+\frac{1}{3}\langle (V^i_i)^2-V^i_jV^j_i\rangle+\frac{1}{3}\langle NN^{|i}{}_{|i}+\dot{N}K^i{}_i\rangle. \tag{17}$$

We assume the matter of the universe to be irrotational dust. Then, the energy-momentum tensor is written

$$T^{\mu\nu}=\rho u^\mu u^\nu, \tag{18}$$

where $u^\mu$ is the four velocity of the fluid flow.

Up to this point, the treatment is fully general and exact. In order to evaluate the R.H.S. of Eqs. (16) and (17), we need the actual metric of the inhomogeneous universe. We can proceed further either with an exact analytic approach employing a class of exact inhomogeneous cosmological solutions or with an approximate treatment of the inhomogeneous metric obtained perturbatively. In the comoving synchronous gauge, i.e., that with $N=1, u^\mu=(1,0,0,0)$, an exact analytical treatment of Eqs. (16) and (17) has been carried out by one of the present authors.[6),7)] It has been shown that there exists a class of exact inhomogeneous solutions that are, nevertheless, homogeneous and isotropic on average, with no backreaction to

the cosmic expansion. Approximate approaches in the form of perturbative analyses have also been performed in several gauges, and some modifications have been reported with apparent discrepancies among those works.[5),10)] In order to clear up this confusion, in the following, we re-examine the approximate approach and solve Eqs. (16) and (17) perturbatively by iteration.

Let us start from the Einstein-de Sitter background as the zeroth-order solution, for simplicity. Next, we obtain the first-order inhomogeneous solution from the linearized Einstein equation. It is sufficient to consider the cosmological post-Newtonian metric at linear order,[3),4),5),11)]

$$ds^2 = -\left(1 + 2\phi(\boldsymbol{x})\right) dt^2 + a^2 \left(1 - 2\phi(\boldsymbol{x})\right) \delta_{ij}\, dx^i dx^j, \tag{19}$$

where $\delta_{ij}$ denotes the Kronecker delta. From the Einstein equation at linear order, we obtain

$$\phi_{,ii} = \frac{3}{2}\dot{a}^2 \left( \frac{\rho - \rho_b}{\rho_b} + 2\phi \right), \tag{20}$$

$$v^i \equiv \frac{u^i}{u^0} = -\frac{2}{3a\dot{a}}\phi_{,i}, \tag{21}$$

where $\rho_b$ is the background density, and contraction has been taken with $\delta_{ij}$. Note that Eq. (19), together with Eqs. (20) and (21), represents the first-order solution of the Einstein equation in the Newtonian gauge.

Once we obtain the first-order solution, we can perform the iteration to solve Eqs. (16) and (17) at the next order. Using the first-order solution to the R.H.S. of Eqs. (16) and (17), and retaining up to quadratic order in $\phi$, the averaged Einstein equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\langle T_{00}\rangle + \frac{1}{a^2}\langle \phi_{,i}\phi_{,i}\rangle, \tag{22}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\langle T_{00} + \rho_b a^2 v^2\rangle - \frac{1}{3a^2}\langle \phi_{,i}\phi_{,i}\rangle, \tag{23}$$

where $v^2 = \delta_{ij} v^i v^j$.

Equation (22) seems to indicate that the nonlinear backreaction expressed by the second term on the R.H.S. might *increase* the expansion rate. However, this is not the case. In order to obtain the correct result, we have to determine the mean energy density $\bar{\rho}$ of dust matter. Most previous works do not give careful consideration of this point. The mean density must satisfy

$$\dot{\bar{\rho}} + 3\frac{\dot{a}}{a}\bar{\rho} = 0. \tag{24}$$

Without this consideration, the backreaction cannot be properly related to a deviation from the Hubble expansion driven by the mean density. Clearly, $\langle T_{00}\rangle$ does not act as $\bar{\rho}$.

In order to guarantee that $\bar{\rho}$ satisfies Eq. (24), we have to define the mean density as

$$\bar{\rho} \equiv \langle T_{00}\rangle + \rho_b a^2 \langle v^2\rangle + \frac{1}{4\pi G a^2}\langle \phi_{,i}\phi_{,i}\rangle = \langle T_{00}\rangle + \frac{5}{12\pi G a^2}\langle \phi_{,i}\phi_{,i}\rangle, \tag{25}$$

where we have used Eq. (21) to rewrite the term containing $\langle v^2 \rangle$ as that with $\langle \phi_{,i}\phi_{,i} \rangle$. The quantity $\bar{\rho}$ is uniquely determined in this form under the following conditions: 1) it is expressed as a linear combination of the terms appearing on the R.H.S. of Eqs. (16) and (17); 2) it satisfies Eq. (24). If these conditions are satisfied, then Eqs. (22) and (23) can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho} - \frac{1}{9a^2}\langle \phi_{,i}\phi_{,i} \rangle, \tag{26}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho}. \tag{27}$$

It is thus seen that we can safely call the second term on the R.H.S. of Eq. (26) the *nonlinear backreaction*.

Equations (26) and (27) reveal the following points.

- The nonlinear backreaction neither accelerates nor decelerates the cosmic expansion. In other words, the cosmic acceleration $\ddot{a}/a$ is determined merely by the mean density.
- The nonlinear backreaction reduces the expansion rate $\dot{a}/a$.
- The nonlinear backreaction is proportional to $a^{-2}$ in the averaged Friedmann equation. Hence, it behaves as a (small) positive curvature term in the Friedmann model. This correction might be measured in future space observations, such as a Planck mission.

It should be noted that Eqs. (26) and (27) do not rely on averaging procedures, though the explicit form of the mean density does. For instance, one may simply use $\langle A \rangle \equiv V^{-1}\int A d^3x$.[4)] Changes induced by using a different averaging scheme occur simultaneously only inside $\bar{\rho}$ of Eqs. (26) and (27). Furthermore, the nonlinear backreaction term is invariant to second order with respect to the choice of the averaging procedure.

It is worthwhile mentioning gauge issues in the backreaction problem. One may wonder what happens under a different gauge condition. The comoving synchronous (CS) gauge has been employed to investigate the influence of the backreaction on quantities used to determine the age of the universe[10)] based on the relativistic version of a Zeldovich-type approximation.[8),9)] Using the CS gauge, one can obtain equations similar to Eqs. (26) and (27). Hence, we reach the conclusion that there is no change in the cosmic acceleration, and the averaged Hubble equation has a positive spatial curvature as a correction. Therefore, it is strongly suggested that the above conclusion is valid for any choice of the gauge. In practice, only the post-Newtonian-type and CS gauges have been employed in previous studies of the nonlinear backreaction problem, as far as the present authors are aware. With this in mind, our conclusion should be considered seriously.

Finally, we make a few comments on related works that have appeared recently. In §3 of Ref. 22), Ishibashi and Wald give an argument concerning the effect of the backreaction, employing an averaging. In spite of their statement that they "point out that our universe appears to be described very accurately on all scales by a Newtonianly perturbed FLRW metric", they restrict their analysis to the comoving

synchronous gauge. They only consider the "smallness" of possible nonlinear effects in the Newtonian gauge. With regard to this point, this Letter gives a new result, which shows explicitly that no cosmic acceleration occurs as a result of the nonlinear backreaction of a Newtonianly perturbed FLRW metric. Hirata and Seljak[23] also reached a negative conclusion concerning the accelerating expansion due to superhorizon cosmological perturbations. Their argument, however, does not treat nonlinear effects due to local (subhorizon) inhomogeneities through averaging, as does the analysis carried out in this Letter.

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